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## THREE PAPERS ON NUMERICAL METHODS OF STRESS ANALYSIS

By

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BOUNDS AND CONVERGENCE OF RELAXATION  
AND ITERATION PROCEDURES

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# BOUNDS AND CONVERGENCE OF RELAXATION AND ITERATION PROCEDURES

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## ABSTRACT

This paper discusses a number of general concepts relating to relaxation procedures, methods of "steepest descent," and iteration procedures. The relationship between these is indicated for certain systematic ways in which relaxation patterns may be developed. An extrapolation procedure is developed for problems in which the successive approximations technique, in its usual form, diverges. The procedure is a generalization of an observation by Hartree and others that in many cases such divergent problems can be treated by considering the divergence to be a geometric one. The study of upper and lower bounds to the errors in relaxation procedures is a second major part of the paper. General procedures are developed for determining the maximum and minimum errors in any of the quantities for which bounds to an influence function can be determined. Use of the theorems derived makes it possible, for example, to estimate the error due to all sources including "round off" at any stage in the numerical solution of Laplace's or Poisson's equation.

## INTRODUCTION

### 1. Summary

This paper describes certain formal procedures for the numerical solution of problems which can be stated in the form of a set of simultaneous linear algebraic equations, although some of the applications extend beyond these limitations. Consideration is given to a method for investigating upper and lower bounds to the numerical values of the solution obtained at any stage. The closeness of the bounds depends upon the choice of a correction pattern, but general rules for selecting the best pattern cannot yet be given.

An investigation is made of the close relationship between extrapolation processes which can be used to speed up the convergence of iterative procedures and "methods of steepest descent" which are used in formal relaxation procedures. Although the convergence of these procedures is slower than for methods which involve the continual exercise of judgment, it seems possible to code the formal methods for use with automatic digital computers.

### 2. General Concepts, Definitions, and Notation

It is assumed that the reader is familiar with the more common details of relaxation and iteration procedures. We consider the set of equations of which the following is a particular equation:

$$\sum_{k=1}^n a_{ik} x_k + b_i = r_i \quad [1]$$

In this equation,  $x_i$  may be considered as the deflection at a point  $i$ ,  $-b_i$  as the load at the point, and  $r_i$  as a residual which becomes zero if all the values of  $x$  are chosen correctly.

It is convenient to deal with a correction pattern of deflections  $u_i$  which produces a change in the residuals of magnitude  $p_i$ ,

$$\sum_{k=1}^n a_{ik} u_k = p_i \quad [2]$$

The quantity  $x + u$  may be a better approximation, in general, to the solution than is  $x$ . A better approximation can always be obtained with the quantity

$$x + tu \quad [3]$$

which involves the parameter or multiplier  $t$ . The solution of the equations corresponding to  $r = 0$  is given by the addition of  $b$  to  $x$ , and is defined as  $\phi$  in the relation

$$\phi_i = x_i + b_i \quad [4]$$

The ordinary iteration procedure is essentially the determination of a new set of values  $x'$  from a given set  $x$  by means of the following scheme. In the  $i$ th equation of [1], set  $r_i = 0$ , and take all  $x_k$  as given except for  $k = i$ . Choose  $x'_i$  so as to satisfy the equation. Repeat for each equation in [1]. In general,  $x'_i$  will be different from  $x_i$ ; in fact

$$x'_i = x_i - r_i / a_{ii}$$

This can be rewritten as follows:

$$x'_i = -\frac{1}{a_{ii}} \left[ \sum_{k=1}^n a_{ik} (1 - \delta_{ik}) x_k + b_i \right] \quad [5]$$

in which

$$\begin{aligned} \delta_{ik} &= 0 & \text{for } i \neq k, \\ \delta_{ik} &= 1 & \text{for } i = k. \end{aligned}$$

We shall choose to write the relation

$$x'_i = x_i + u_i \quad [6]$$

to indicate the change in  $x$  produced by the single step in the iteration process. Similarly, if we start with a value  $y$  instead of  $x$ , we obtain a new value

$$y' = y + v. \quad [7]$$

Other notation will be defined when it is introduced.

### UPPER AND LOWER BOUNDS

#### 3. General Relations

The solution of [1], with  $r = 0$ , can be stated as

$$\phi_i = x_i + b_i = - \sum_k A_{ik} b_k. \quad [8]$$

Now, suppose some quantity  $Q_n$  is desired which is a linear function of  $\phi$ , as follows:

$$Q_n = \sum_i a_{ni} \phi_i \quad [9]$$

Then, let us define a new quantity  $\beta_i$  such that

$$Q_n = - \sum_i a_{ni} \sum_k A_{ik} b_k = - \sum_k \beta_{nk} b_k \quad [10] \text{ and if}$$

in which

$$\beta_{nk} = - \sum_i a_{ni} A_{ik} \quad [11]$$

The quantities  $\beta_{nk}$  can be considered as an influence function for the quantity  $Q_n$  for unit values of the load  $-b_k$ .

In the same way that  $Q$  corresponds to the true solution  $\phi$ , let us define a quantity  $M$  corresponding to the approximate solution  $x$ ,

$$M_n = \sum_i a_{ni} x_i$$

or

$$M_n = - \sum_k \beta_{nk} (b_k - r_k) \quad [12]$$

This follows from the fact that the deflections  $x_i$  corresponds to loads  $-b_i + r_i$  in the same way that the deflections  $\phi_i$  correspond to loads  $-b_i$ .

It follows immediately that the difference between  $Q$  and  $M$  can be expressed as

$$M_n - Q_n = \sum_k \beta_{nk} r_k \quad [13]$$

If this difference is positive, then  $Q$  is definitely less than  $M$ , and if it is negative,  $Q$  is greater than  $M$ . It follows, therefore, that if we know the values of  $\beta$ , then for any set of values  $x$  which implies a set of values of  $r$ , we could determine whether  $M$  is greater than or less than  $Q$ . But we do not know  $\beta$ . However, under certain conditions we can determine bounds or upper and lower limits to  $\beta$  readily enough.

Let us designate those values which are upper bounds by the symbol  $\beta^a$  and lower bounds by  $\beta^b$ . That is, whether  $\beta$  is positive or negative, the relation holds, that

$$\beta^b \leq \beta \leq \beta^a \quad [14]$$

Next, we separate from  $r_k$  those values which are positive from those which are negative and designate them respectively by  $r_k^+$  and  $r_k^-$ . In investigating the sign of the right hand side of [13], we must use separately the upper and lower limits of  $\beta$ , as follows:

If

$$\sum_k (\beta_{nk}^b r_k^+ + \beta_{nk}^a r_k^-) \geq 0 \quad [15]$$

then

$$M_n \geq Q_n \quad \text{and} \quad M_n = M_n^a$$

and if

$$\sum_k (\beta_{nk}^a r_k^+ + \beta_{nk}^b r_k^-) \leq 0 \quad [16]$$

then

$$M_n \leq Q_n \quad \text{and} \quad M_n = M_n^b.$$

If these inequalities are not satisfied, the criterion fails.

#### 4. Use of Relaxation Patterns

In the study of bounds it will rarely occur that the criteria of equations [15] and [16] will be satisfied for a particular set of residuals and given values of upper and lower limits to  $\beta$ . We shall, therefore, consider ways of adding a proportion of a correction pattern  $u$  to  $x$  in order to arrive at a useful result. The pattern  $u$  corresponds to changes in residuals  $p$ , in accordance with equation [2]. We seek a multiplier  $t$  such that the solution  $x + tu$  corresponds to residuals  $r + tp$  which give us an upper or lower bound for  $Q$ , according to [15] or [16].

For any range in the values of  $t$  for which  $r_k + tp_k$  does not change in sign, we can derive from equation [15] the relation

$$\sum_k \beta_{nk}^b (r_k + tp_k)^+ + \sum_k \beta_{nk}^a (r_k + tp_k)^- \geq 0 \quad [17]$$

The quantities in parentheses must be ordered according to the sign of their sum, as indicated. Now, those values of  $r$  and  $p$  which come from  $(r + tp)^+$  are designated  $r_k^{++}$

and  $p^{(+)}$ ; even though some of these values may be negative; and similarly for  $r^{(-)}$  and  $p^{(-)}$  which come from  $(r + tp)$ . Then we can write:

$$t \sum_k \left[ \beta_{nk}^b p_k^{(+)} + \beta_{nk}^a p_k^{(-)} \right] + \sum_k \left[ \beta_{nk}^b r_k^{(+)} + \beta_{nk}^a r_k^{(-)} \right] \geq 0 \quad [13]$$

from which we determine a minimum value of  $t$ :

$$t_{min} = \frac{\sum_k \left[ \beta_{nk}^b r_k^{(+)} + \beta_{nk}^a r_k^{(-)} \right]}{\sum_k \left[ \beta_{nk}^b r_k^{(+)} + \beta_{nk}^a p_k^{(-)} \right]} \quad [19]$$

If the value of  $t$  in [19] requires changes in the sign of any  $r_k + tp_k$ , the corresponding change in the summations must be made. However, by a process involving several trials a value of  $t$  may be obtained which corresponds therefore to the desired upper bound  $M_n^a$ .

A similar expression can readily be written for the value of  $t$ , from equation [16], corresponding to a lower bound  $M_n^b$ . It is clear that the values obtained depend on the patterns used, and in many cases it may not be possible to satisfy the criteria for a given pattern.

In certain cases it is convenient to use as a correction pattern a set of values proportional to  $x$ . This does not require special consideration beyond the remark that when  $u = x$ , the changes in residuals  $p$  are given simply by the equation

$$p_k = r_k - b_k \quad [20]$$

### 5. Special Cases

Certain special cases are of particular interest because of their simplicity and general application. For example, in such problems as the flexure of certain plates or the deflection of membranes it is obvious that a load at any point produces a deflection at every point in the same general direction. For deflections, at least, it is clear that for any point  $n$ , a lower limit to  $\beta$  is a positive number approaching zero, and an upper limit is some positive number  $B$ . It follows directly from equations [15] and [16] that:

$$M = M^a \text{ if all residuals are positive,}$$

and

$$M = M^b \text{ if all residuals are negative.} \quad [21]$$

These conditions are not nearly as complicated as those that might be used to obtain more accurate bounds, but they are extremely useful as a guide to judgment in relaxation solutions for problems where the criteria are applicable.

When all the influence values are positive, a simpler condition than [19] can be derived. We seek a multiplier  $t$  so as to make, for example, every value of total residual positive, i.e.

$$r_i + tp_i \geq 0 \quad [22]$$

Now, let us define  $t_i$  as follows

$$t_i = -r_i/p_i \quad [23]$$

We must consider separately positive and negative values of  $p_i$ . We shall designate the greatest algebraic value of  $t_i$  for positive values of  $p$  as  $T_+^a$  and for negative values of  $p$  as  $T_-^a$ . Similarly we shall designate the minimum algebraic values of  $t_i$  for positive  $p$  as  $T_+^b$  and for negative  $p$  as  $T_-^b$ .

Then, in order that equation [22] be satisfied for all values of  $t$ , the following condition must apply:

$$T_-^b \geq t \geq T_+^a \quad [24]$$

and it is only under these restricted conditions that this relation can be satisfied.

Similarly, in order that the following relation hold:

$$r_i + tp_i \leq 0 \quad [25]$$

the required condition is,

$$T_+^b \geq t \geq T_-^a \quad [26]$$

Another special case of considerable interest concerns bounds for the volume under a surface the ordinates to which must satisfy given conditions. Numerically, the volume can be expressed as a set of multipliers, essentially nearly equal, times the individual ordinates, or as a consequence, as a constant times the sum of the ordinates. In other words, the values of  $a$  in equations [9] and [11] are unity.

It follows, therefore, that

$$Q = \sum_i \phi_i = \sum_i x_i + \sum_i b_i \quad [27]$$

But in cases where  $x$  or  $u$  is a deflection, and  $r$  or  $p$  is a load, we can write the relation:

$$b_i = - \sum_k r_k u_i^{r_k=1} \quad [28]$$

(where  $u_i^{r_k=1}$  means the deflection at  $i$  due to a unit load at  $k$ ). We can regard the differences between the true deflections  $\phi$  and  $x$  as caused by the residuals  $r$ .

However, because of Maxwell's theorem of reciprocal deflections, [28] can be rewritten as follows:

$$b_i = - \sum_k r_k u_k^{r_i=1} \quad [29]$$

Then we can derive the result:

$$\sum_i b_i = - \sum_i \sum_k r_k u_k^{r_i=1} = - \sum_k r_k \sum_i u_k^{r_i=1} \quad [30]$$

But the last summation is merely the deflection at  $k$  due to unit loads at all points  $i$ , and we can designate this by the symbol  $\bar{u}_k$ .

As a result we have, if we designate  $\sum_i x_i$  by the symbol  $M$ :

$$M - Q = \sum_k \bar{u}_k r_k \quad [31]$$

where  $\bar{u}_k$  is the deflection pattern produced by a uniform load of unity at every point. But this is precisely the same equation as [13] with  $\beta_{nk}$  replaced by  $\bar{u}_k$ ; consequently the same comments apply, and the criteria expressed by equations [15] and [16] can be used to determine whether  $M$  is an upper or lower bound, if we can obtain bounds for  $\bar{u}$ . In general this is not particularly difficult to do for such problems as are encountered with membranes and plates.

Since the determination of the torsion constant for a prismatic bar requires the calculation of a volume, bounds to the torsion constant can usually be determined quite accurately from equations [31], [15] and [16]. These give a much closer limit than can be obtained from the condition requiring all residuals to be of the same sign.

#### 6. Bounds for Influence Function

In the use of the procedure described herein it is necessary to estimate upper and lower bounds for the influence functions in the equations. The simplest influence function is that for the deflection at a particular point  $n$ , for which the values of  $\alpha_{ni}$  in equations [9] and [11] are zero except for  $i = n$ . This influence function can be obtained qualitatively in most physical problems quite readily as the deflection function for a unit load at point  $n$ . However, no simple rule for the determination of the influence function can be given for problems involving arbitrary systems of equations.

In physical problems which deal with loads and deflections, use can be made of a simple theorem to restate the problem of determining an influence function in terms of the problem of finding deflections for given loads. This theorem is derived in a previous paper by the author and is stated as follows:

Where a particular effect  $Q_n$  is a linear function of the deflections  $x_i$  at points  $i$ , as in the equation  $Q_n = \sum A_{ni} x_i$ ,

and if deflections and loads are linearly related, the influence on  $Q_n$  of a unit load at any point  $k$  is obtained as the deflection at the point  $k$  due to loads of magnitude  $A_{ni}$  applied at the points  $i$ .

In problems to which this theorem is applicable, the bounds for the influence functions can be obtained in the same way as bounds for any other problem.

Consider, for example, the problem of determining bounds to the moment in a plate at a specific point due to a given distribution of loading on the plate. In solving the problem, one has determined deflections and residuals for the given loading, and presumably one has computed a number of relaxation patterns. Now the influence function for a moment can be determined as the deflection of the plate due to a system of loads grouped near the point considered, some upward and others downward. A quick estimate can often be made directly of upper and lower

bounds for the influence function either by use of judgment or by a rough calculation and several cycles of relaxation. It appears offhand to be necessary to go through the problem several times, but this is hardly warranted, since the trouble involved in closely delimiting the bounds is great enough to warrant instead the calculation of more accurate figures with additional cycles of relaxation or iteration.

### EXTRAPOLATION WITH ITERATIVE PROCEDURES

#### 7. General Relations

A brief but interesting and valuable discussion of iterative procedures has been given by Hartree<sup>1</sup> in which he states a process of improving or of essentially hastening the convergence of a sequence of iterative calculations of numbers. There is described here a generalization of this procedure which applies to sets of numbers, and particularly to iterative solutions of simultaneous equations, although it is applicable to other problems.

Consider a set of numbers  $x_i$  which after one iterative cycle leads to a new set  $x'_i = x_i + u_i$ . A similar set  $y_i$  leads to  $y'_i = y_i + v_i$ .

We seek a set of numbers  $z_i$  leading to  $z'_i$  which is to be closer in general to  $z_i$  than  $x'_i$  or  $y'_i$  is to  $x_i$  or  $y_i$ .

When the system is linear or even approximately linear, we proceed as follows:

$$\begin{aligned} z_i &= x_i + t(y_i - x_i) \\ z'_i &= x'_i + t(y'_i - x'_i) = z_i + u_i + t(v_i - u_i) \end{aligned} \quad [32]$$

Now let us choose to minimize the expression

$$\sum [u_i + t(v_i - u_i)]^2 \quad [33]$$

This leads to the condition

$$t = - \frac{\sum u_i v_i}{\sum v_i^2}, \text{ where } u_i = v_i - u_i \quad [34]$$

We have thus "extrapolated" or "jumped" over several iterations. We can state the relations also for the case of three successive steps in which  $y = x''$ ,  $z = y' = x''$ . Then

$$u = x' - x, \quad w = x'' - 2x' + x \quad [35]$$

and

$$t = - \frac{\sum (x'_i - x_i)(x''_i - 2x'_i + x_i)}{\sum (x''_i - 2x'_i + x_i)^2} \quad [36]$$

The best value for the next iteration is then

$$z_i = x_i + t(x''_i - x_i) \quad [37]$$

When we have only one value of  $i$ , we have individual numbers instead of a set, and equation [37] reduces to the relation

<sup>1</sup>N. M. Newmark, "Note on Calculation of Influence Surfaces in Plates by Use of Difference Equations," *Journal of Applied Mechanics*, Vol. 8, No. 2, June 1941, p. A-92.

<sup>2</sup>D. R. Hartree, "Calculating Instruments and Machines," University of Illinois Press, Urbana, 1949, p. 116.



$$z = x - (x' - x)^2 / (x'' - 2x' + x) \quad [38]$$

which is the form given by Hartree for this case.

The use of these relationships effects a marked improvement in convergence. As a matter of fact, the extrapolation leads to convergence even in situations where straightforward iteration diverges rapidly. For linear systems the form of the result indicates that convergence may always be expected formally. However there are cases where even this procedure converges too slowly to be practically useful.

### 8. Relation of Extrapolation to Relaxation

An interesting relation between extrapolation and iteration can be derived. If we use formal iterative procedures in a set of linear equations, as in equations [5] and [6], we find that we can relate  $u_i$  and  $r_i$  in the following way:

$$u_i = -r_i / a_{ii} \quad [39]$$

Now if we take a relaxation pattern  $w_i$  and complete the change in residuals  $p_i$  from equation [2], we can show that  $w_i$  in equation [35] can be written as:

$$w_i = -p_i / a_{ii} \quad [40]$$

If instead of equation [33] we had chosen to minimize the quantity

$$\sum m_i [x_i + t w_i]^2,$$

we would find the condition:

$$t = - \frac{\sum m_i u_i w_i}{\sum m_i w_i^2} \quad [41]$$

In these relations  $m_i$  is an arbitrary weight factor. We may take  $m_i$  as  $a_{ii}^{-1}$ . If we do, we find:

$$t = - \frac{\sum r_i p_i}{\sum p_i^2} \quad [42]$$

It will be shown later that these equations are nearly the same as those derived by use of a slight modification of Temple's criterion for a "steepest descent" process in relaxation.

## FORMAL RELAXATION PROCEDURES

### 9. General Relations for Steepest Descent Methods

In this paper consideration is given only to formal procedures. Several variations of steepest descent methods have been described in the literature<sup>1</sup>, and the general failings of these procedures have also been discussed by

Hartree and Courant. No more will be said here on these points which are still under study.

If in a set of equations such as [1] we seek to correct or better the approximate values  $x$  by the addition of  $tu$  where  $u$  is a correction pattern, we have several options. One which seems reasonable is to choose  $u$  along the gradient of the function

$$\sum m_i r_i^2 = F \quad [43]$$

and to select  $t$  so as to minimize the quantity  $\sum m_i (r_i + t p_i)^2$ .

This leads to the following process:

(a) For the given values of  $x$  compute  $r$  from equations [1].

(b) Operate on the values of  $r$  as if they were a corrective pattern, to determine  $u$ . In this operation, however, transpose the coefficients of the matrix to use  $a_{ki}$  instead of  $a_{ik}$ . That is

$$\sum_k a_{ki} r_k m_k = u_i \quad [44]$$

In many sets of equations  $a_{ki} = a_{ik}$ , or the matrix is symmetric.

(c) Take  $u$  as a correction pattern and determine  $p$  from equation [2]

$$\sum a_{ik} u_k = p_i$$

(d) Then determine  $t$  as follows:

$$t = - \frac{\sum m_i r_i p_i}{\sum m_i p_i^2} \quad [45]$$

(e) The new value of  $x$  is now  $x + t u$ .

The procedure described requires two sets of operations to determine the correction pattern, and it is therefore unwieldy.

In Temple's procedure the quantity that is minimized is the potential energy of the system, which for a symmetrical matrix can be expressed in the form:

$$U = \frac{1}{2} \sum_i \sum_k a_{ik} x_i x_k + \sum_k b_k x_k \quad [46]$$

The procedure turns out to be as follows:

(a) compute  $r$  for the given values of  $x$ .

(b) take  $u_i = r_i$  as the correction pattern.

(c) compute  $p_i$  from equation [2].

(d) compute  $t$  from the relation:

$$t = - \frac{\sum r_i^2}{\sum r_i p_i} \quad [47]$$

A slight modification of Temple's procedure appears to be better in most cases, and is applicable to non-symmetric matrices. This involves arbitrarily taking alternately two types of correction pattern.

One type is

<sup>1</sup>G. Temple, "The General Theory of Relaxation Methods Applied to Linear Systems," Proc. Roy. Soc., London, Ser. A, Vol. 166, 1939, p. 476-500.

R. Courant, "On a Method for the Solution of Boundary-Value Problems," Theodore von Karman Anniversary Volume, California Institute of Technology, Pasadena, 1941.

D. R. Hartree, "Experimental Arithmetic," Eureka, March, 1948, p. 13.

$$u_i = r_i \quad [48]$$

and the other is a pattern proportional to the deflections, i.e.

$$u_i = x_i \quad [49]$$

In either case, the quantity to be minimized is

$\sum_i m_i (r_i + t p_i)^2$  with, in most cases,  $m_i = 1$ . In general, the value of  $t$  is

$$t = - \frac{\sum_i m_i r_i p_i}{\sum_i m_i p_i^2}$$

as in equation [45])

It is interesting to note that Temple's criterion, equation [47], is equivalent to minimizing the quantity

$$\frac{1}{t^2} \sum_i (r_i + t p_i)^2 = \sum_i \left( \frac{1}{t} r_i + p_i \right)^2 \quad [50]$$

#### 10. Use of Orthogonality

It can be shown that in general with the steepest descent procedures, the vectors corresponding to successive steps are orthogonal, or nearly orthogonal in successive pairs, but in general they are not orthogonal to other vectors in preceding steps. It is entirely possible, however, to build up a system of orthogonal vectors to eliminate any set of residuals. Since more generality is obtained by considering a weighting factor, it is included in the argument.

Suppose we consider a pattern  $u$  corresponding to changes in residuals  $p$  according to equation [2]. A second pattern  $\bar{v}$  corresponds to changes in residuals  $\bar{q}$ . It is desired to find a pattern  $v$  corresponding to residuals  $q$  such that the following condition is satisfied:

$$\sum_i m_i p_i q_i = 0 \quad [51]$$

We can achieve this result by removing from  $\bar{q}$  and  $\bar{v}$  the portion that is not orthogonal to  $p$  and  $u$ .

$$\text{Let } \bar{v} = v + cu$$

$$\text{Then } \bar{q} = q + cp$$

and

$$\sum_i m_i p_i \bar{q}_i = \sum_i m_i p_i q_i + c \sum_i m_i p_i^2 \quad [52]$$

But the second term in [52] is zero. Therefore we have the result

$$c = \frac{\sum_i m_i p_i \bar{q}_i}{\sum_i m_i p_i^2} \quad [53]$$

and

$$q = \bar{q} - cp, \quad v = \bar{v} - cu. \quad [54]$$

By repeated steps we can build up a system of patterns and changes in residuals which are mutually orthogonal in the sense of equation [51]. We need to determine only as many sets of values as there are unknowns in our equations. Then to solve the equations, we obtain a set of initial residuals  $r$  corresponding to some set of values of  $x$ , and proceed to resolve  $r$  into components corresponding to each of the patterns. The coefficient in the series of components corresponding to  $p^{(n)}$  is obtained as follows:

$$r = \sum_n t_n p^{(n)} \quad [55]$$

$$t_n = - \frac{\sum_i m_i r_i p_i^{(n)}}{\sum_i m_i p_i^{(n)} p_i^{(n)}}$$

The final quantities are then

$$\phi_i = x_i + \sum_n t_n \phi_i^{(n)} \quad [56]$$

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## A NUMERICAL METHOD FOR THE SOLUTION OF PLATE BUCKLING PROBLEMS

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### ABSTRACT

A numerical method is presented for the solution of certain problems concerned with buckling of flat, rectangular plates under conditions for which the usual partial differential equation of equilibrium can be reduced to an ordinary differential equation.

### FUNDAMENTAL EQUATIONS

THE differential equation governing the buckling of a plate is given by Equation [1],

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = -\frac{P_0}{N} \left( p_x \frac{\partial^2 w}{\partial x^2} + p_y \frac{\partial^2 w}{\partial y^2} + 2 p_{xy} \frac{\partial^2 w}{\partial x \partial y} \right) \quad [1]$$

where  $w$  is the deflection,  $N$  is the slab stiffness,  $P_0$  is a parameter proportional to the loading,  $p_x$  and  $p_y$  are the compressive buckling forces per unit of length acting in the  $x$  and  $y$  directions, respectively, relative to the load parameter  $P_0$ , and  $p_{xy}$  is a similar measure of shearing force.

A transformation of this equation into an ordinary differential equation can be made in the special case of a rectangular plate with the following edge conditions and distribution of loading:

- (1) The edges  $x = 0$  and  $x = s$  are hinged.
- (2) The edges  $y = 0$  and  $y = b$  may have any condition of support, not necessarily similar.
- (3) There are no shearing stresses in the plane of the plate parallel to either edge; thus  $p_{xy} = 0$  everywhere in the plate.
- (4) The normal stress  $\sigma_y$  acting in the  $y$ -direction is uniform over the plate.

(5) The normal stress  $\sigma_x$  acting in the  $x$ -direction is a linear function of  $y$ . This condition is necessary to insure a zero value of  $\tau_{xy}$  throughout the plate.

In the following discussion the longitudinal and transverse directions refer to the  $x$  and  $y$  directions, respectively.

For the buckling problems considered, the deflection surface can be represented as follows:

$$w = Y(y) \sin \frac{m\pi x}{s} = Y \sin \frac{\pi x}{a} \quad [2]$$

where  $Y(y)$  is a function of  $y$  only  $a$  is the longitudinal length of one buckle, equal to  $s/m$ , and  $m$  is the number of half sine waves in the longitudinal direction. The length  $a$  must be specified before the solution can be derived.

When Equation [2] is substituted into Equation [1], and if  $p_{xy}$  is set equal to zero Equation [3] is obtained. Each prime symbol denotes one differentiation with respect to  $y$ .

$$Y'''' - 2 \frac{\pi^2}{a^2} Y'' + \frac{\pi^4}{a^4} Y = \frac{P_0}{N} \left( p_x \frac{\pi^2}{a^2} Y - p_y Y'' \right) \quad [3]$$

The scope of this special case is restricted; however, many practical problems can be simplified to fall in this category and nearly all available "exact" solutions embody the restricted conditions described. An excellent discussion of the classical methods and a fairly complete compilation of available solutions is given by Timoshenko.<sup>1</sup>

This paper presents a numerical method of solution of Equation [3] which is a combination of two well-known procedures. The Stodola-Vianello method of successive approximations is combined with the numerical procedure of integration published by Newmark.<sup>2</sup>

Recently Stüssi<sup>3</sup> has published an article in which this same class of problems is solved by a numerical method. However, there is only slight similarity in the method of solution proposed by Stüssi and the method presented herein.

### EDGE CONDITIONS

Within the scope of the method, a number of different conditions of support on the longitudinal edges can be treated. Of these only the following three common edge conditions are described in detail:

<sup>1</sup>"Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, 1936.

<sup>2</sup>"Numerical Procedure for Computing Deflections, Moments, and Buckling Loads," by N. M. Newmark, *Transactions ASCE*, Vol. 108, 1943, p. 1161.

<sup>3</sup>"Berechnung Der Beanspruchungen Gedrückter Rechteckplatten," by F. Stüssi, *Publications, Int. Assoc. of Bridge and Structural Engineers*, V. 1, 8, 1947, p. 237.

(a) Hinged Longitudinal Edge

Along a hinged edge  $y = A$ , the deflections and the moments (hence curvatures) normal to the edge are zero. Therefore:

$$Y_A = 0 \quad \text{and} \quad Y_A'' = 0$$

(b) Fixed Longitudinal Edge

Along a fixed edge  $y = A$  the deflections and the slopes normal to the edge are zero. Hence

$$Y_A = 0 \quad \text{and} \quad Y_A' = 0$$

(c) Free Longitudinal Edge

Along a free edge the reactions and the moments normal to the edge are zero. Hence

$$\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} = 0$$

and

$$\frac{\partial^2 w}{\partial y^2} + (2 - \mu) \frac{\partial^2 w}{\partial x^2 \partial y} = 0$$

where  $\mu$  = Poisson's ratio. When the deflections defined by Equation [2] are substituted into these last conditions, the boundary requirements are transformed into the following:

$$Y_A = \frac{b^2}{\mu \pi^2 / n^2} Y_A''$$

and

$$Y_A' = \frac{b^2}{(2 - \mu) \pi^2 / n^2} Y_A'''$$

where  $a = nb$ .

### GENERAL PROCEDURE

Equation [3] may be restated, for convenience in the calculations, as follows:

$$Y'''' - 2 \frac{\pi^2}{n^2} \frac{1}{b^2} Y'' + \frac{\pi^4}{n^4} \frac{1}{b^4} Y = K \left( p_x \frac{\pi^2}{n^2} \frac{1}{b^2} Y - p_y \frac{1}{b^2} Y'' \right) \quad [4]$$

in which  $K = P_0 b^2 / N$  and  $a = nb$ . The quantity  $K$  is a dimensionless factor which determines the magnitude of the given pattern of buckling forces. The plate width  $b$  is divided into any desired number of segments of equal length  $b$ . Then the length of the plate  $a$  is expressed as  $n$  segments of length  $b$ . It is not necessary that  $n$  be an integer.

Using the notation

$$Y_R'''' = - \frac{\pi^4}{n^4} \frac{1}{b^4} Y + 2 \frac{\pi^2}{n^2} \frac{1}{b^2} Y'' \quad [5a]$$

and

$$Y_D'''' = p_x \frac{\pi^2}{n^2} \frac{1}{b^2} Y - p_y \frac{1}{b^2} Y'' \quad [5b]$$

one obtains Equation [6] from Equation [4].

$$Y'''' = Y_R'''' + K Y_D'''' \quad [6]$$

The solution is obtained by the following procedure:

(a) A set of deflections  $Y_a$  is assumed, one value at each node point.

(b) With the assumed deflections, the right hand sides of Equations [5a] and [5b] are evaluated. These equations are then solved numerically to yield the two component deflection functions  $Y_R$  and  $Y_D$ . The numerical process used is described later. Each deflection function must separately satisfy the boundary requirements; then the sum of the two functions  $Y$  will also satisfy the boundary conditions for any value of  $K$ .

(c) The deflections  $Y_a$  and  $Y = Y_R + K Y_D$  are now compared. If they can be made identically equal for a particular value of  $K (= P_0 b^2 / N)$ , then this value of  $K$  defines the magnitude of the critical buckling load, and  $Y_a$  or  $Y$  is the corresponding configuration. If the assumed deflections cannot be made equal to the computed deflections, new values of  $Y_a$  must be assumed and steps (a) to (c) must be repeated until a desired measure of agreement is reached.

When the comparison indicates that the assumed set of deflections is not correct, a new and more accurate set of deflections can be obtained from the calculations. This improved deflection configuration can then be used as the assumed set of deflections in the next cycle of computations. The new configuration is computed for the value of  $K$  which makes  $Y_R + K Y_D$  most nearly equal to  $Y_a$ .

The numerical solution depends upon the length  $a$ . The procedure will not indicate the proper number of half sine waves in the longitudinal direction to yield the lowest critical load in any given problem. If the proper value of  $a$  is not obvious, trials with several values of  $m$  are necessary.

The functions  $Y_R''''$  and  $Y_D''''$ , expressed by Equations [5a] and [5b], cannot be directly evaluated because the required values of  $Y_a$  are not known. Therefore, each of these functions is split into two parts, as follows:

$$Y_R'''' = {}_1Y_R'''' + {}_2Y_R''''$$

where

$${}_1Y_R'''' = - \frac{\pi^4}{n^4} \frac{1}{b^4} Y$$

$${}_2Y_R'''' = 2 \frac{\pi^2}{n^2} \frac{1}{b^2} Y''$$

and

$$Y_D'''' = {}_1Y_D'''' + {}_2Y_D''''$$

where

$${}_1Y_D'''' = p_x \frac{\pi^2}{n^2} \frac{1}{h^4} Y_a$$

$${}_2Y_D'''' = -p_y \frac{1}{b^2} Y_a''$$

The general expressions for the third derivatives, found by integration, are given in Equations [7a] and [7b].

$$Y_R''' = {}_1Y_R''' + {}_2Y_R''' = \int {}_1Y_R'''' dy + 2 \frac{\pi^2}{n^2} \frac{1}{b^2} Y_a' \quad [7a]$$

$$Y_D''' = {}_1Y_D''' + {}_2Y_D''' = \int {}_1Y_D'''' dy - p_y \frac{1}{b^2} Y_a' \quad [7b]$$

The third derivatives of the component deflection functions cannot be completely evaluated unless values of  $Y_a'$  are known. The total values are not needed except at a free edge or an edge elastically restrained from deflection.

The formulas for the second derivatives are as follows:

$$Y_R'' = {}_1Y_R'' + {}_2Y_R'' = \int dy \int {}_1Y_R'''' dy + 2 \frac{\pi^2}{n^2} \frac{1}{b^2} Y_a$$

$$Y_D'' = {}_1Y_D'' + {}_2Y_D'' = \int dy \int {}_1Y_D'''' dy - p_y \frac{1}{b^2} Y_a$$

Note that unlike the third and fourth derivatives, the second derivatives may be directly evaluated from a knowledge of only the assumed deflections  $Y_a$ . The component deflection functions,  $Y_R$  and  $Y_D$ , may now be determined by two straightforward integrations of the total second derivatives,  $Y_R''$  and  $Y_D''$ .

A symbolic representation of the procedure is given in Fig. 1. This figure illustrates the notation used and gives a picture of the plan of the solution. The integration symbols denote numerical integration. In step 2 of Fig. 1 the quantities  ${}_1Y_R''''$  or  ${}_1Y_D''''$  are evaluated. Steps 3 and 4

of Fig. 1 each indicate one integration of the component evaluated in step 2. The total values of  $Y'''$  are not determined in Fig. 1; if desired they may be evaluated in accordance with Equations [7a] and [7b]. The total values of  $Y''$  are computed in step 6. Steps 7 and 8 are straightforward integrations. It can be seen from Fig. 1 that the operations used to determine the two component deflection functions  $Y_R$  and  $Y_D$  are identical.

In the common factor column of Fig. 1 the factor for  ${}_1Y''''$  is  $c/h^4$ , where  $c$  is any arbitrary unit of deflection. However, during the integrations the quantities are multiplied several times by  $b$ , so that when the operations are completed the factor for  $Y$  is the same as that of the assumed deflection. Thus the factors for all deflections are consistent. It is not necessary to evaluate the increment

TABLE OF OPERATIONS

No. of Step	Designation of Step	Computation	Common Factors
Calculation of $Y_R$			
1	Assumed Deflection	$Y_a$	$c$
2	${}_1Y_R''''$	$- \frac{\pi^2}{n^2} Y_a$	$c/h^4$
3	${}_1Y_R'''$	$\int {}_1Y_R'''' dy$	$c/h^3$
4	${}_1Y_R''$	$\iint {}_1Y_R'''' dy dy$	$c/h^2$
5	${}_2Y_R''$	$2 \frac{\pi^2}{n^2} Y_a$	$c/h^2$
6	$Y_R''$	${}_1Y_R'' + {}_2Y_R''$	$c/h^2$
7	$Y_R'$	$\int Y_R'' dy$	$c/h$
8	$Y_R$	$\iint Y_R'' dy dy$	$c$
Calculation of $Y_D$			
1	Assumed Deflection	$Y_a$	$c$
2	${}_1Y_D''''$	$p_x \frac{\pi^2}{n^2} Y_a$	$c/h^4$
3	${}_1Y_D'''$	$\int {}_1Y_D'''' dy$	$c/h^3$
4	${}_1Y_D''$	$\iint {}_1Y_D'''' dy dy$	$c/h^2$
5	${}_2Y_D''$	$- p_y Y_a$	$c/h^2$
6	$Y_D''$	${}_1Y_D'' + {}_2Y_D''$	$c/h^2$
7	$Y_D'$	$\int Y_D'' dy$	$c/h$
8	$Y_D$	$\iint Y_D'' dy dy$	$c$

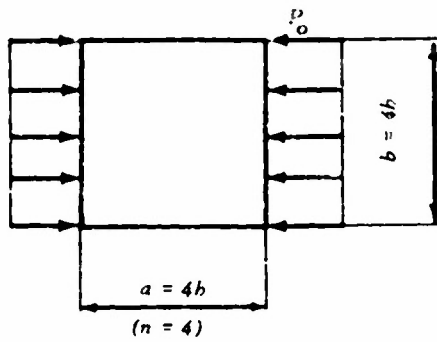
FIG. 1.

length,  $b$ , in the course of the operation.

The various corrections required to satisfy boundary conditions are not indicated in Fig. 1, but are discussed later.

Consider, for example, the calculation of the critical buckling load for a square plate, hinged on four edges, with uniformly distributed compressive forces which act in the longitudinal direction. The calculations for an assumed sinusoidal deflection configuration  $Y_a$  are given in Problem 1. The plate width  $b$  has been divided into four segments of equal length  $b$ . Since the plate is square, the length of the plate is also equal to four increments  $b$ ; hence the value of  $n$  for this problem is 4. Note that the constants to be used in the numerical solution are given to the right of the figure.

# PROBLEM 1



$$\frac{\pi^2}{n^2} = 0.617$$

$$2 \frac{\pi^2}{n^2} = 1.234$$

$$\frac{\pi^4}{n^4} = 0.381$$

All Edges Hinged

Step No.	Description				Common Factors
Calculation of $Y_R$					
1	$Y_a$	0	7.07	10.00	c
2	${}_1Y_R''''$	0	-2.59	3.81	$c/b^4$
	$\bar{Y}_R''''$		-2.56	3.62	$c/b^3$
3	${}_1Y_R'''$	4.37	1.81		$c/b^3$
4	${}_1Y_R''$	0	4.37	6.18	$c/b^2$
5	${}_2Y_R''$	0	8.72	12.34	$c/b^2$
6	$Y_R''$	0	13.09	18.52	$c/b^2$
	$\bar{Y}_R''$		12.45	17.62	$c/b$
7	$Y_R'$	-21.26	-8.81		$c/b$
8	$Y_R$	0	-21.26	-30.07	c
Calculation of $Y_D$					
1	$Y_a$	0	7.07	10.00	c
2	${}_1Y_D'''' = Y_D''''$	0	4.36	6.17	$c/b^4$
	$\bar{Y}_D''''$		4.15	5.86	$c/b^3$
3	$Y_D'''$	-7.08	-2.93		$c/b^3$
4,5,6	$Y_D''$	0	-7.08	-10.01	$c/b^2$
	$\bar{Y}_D''$		-6.73	-9.52	$c/b$
7	$Y_D'$	11.49	4.76		$c/b$
8	$Y_D$	0	11.49	16.25	c
$K = \frac{\sum (Y_a - Y_R)}{\sum Y_D} = \frac{24.14 + 72.59}{39.23} = 2.466, \text{ from Equation (8)}$					
	$Y = Y_R + KY_D$	0	7.07	10.00	c
$P_0 = \bar{P}_{cr} = 2.466 N/b^2 = 39.46 N/b^2$					

The presentation of the first illustrative problem follows the operations indicated in Fig. 1. Almost every line of calculations has a step number which corresponds to the equivalent line in the operations table of Fig. 1. Between steps 2 and 3 and steps 6 and 7 are lines of computations which are not identified with the operations table. These lines contain the calculations of equivalent concentrations, a basic feature of the numerical summation method as given by Newmark. A bar above the symbol for a distributed derivative denotes an equivalent concentrated derivative. Since the functions  $Y''''$  and  $Y''$  are smooth curves, the equivalent concentrations are calculated by the parabolic formulas given in Fig. 2. The values of the odd derivatives,  $Y'''$  and  $Y'$ , may be considered as average values computed for the interval between node points. For the sake of brevity the word "average" is omitted. Values can be computed at the node points, if desired.

On the extreme right hand side of the computations in Problem 1 is a column of common factors. These factors agree with those in the operations table and the changes in them from one step to another are consistent with the numerical summation method.

At the top of the calculation is a diagram of the transverse section of the plate. This diagram indicates the conditions on the longitudinal edges and defines the position of the node points at which the calculations are being performed.

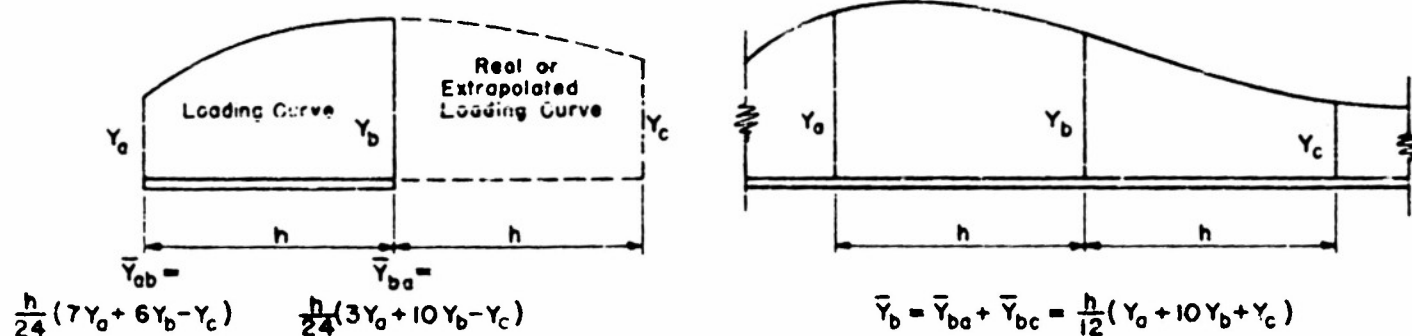
Only half of the plate is considered in the computation since the structure and the deflections are symmetrical about the center line. The derivatives of even order are symmetrical about the center line and the derivatives of odd order are anti-symmetrical about the center line. The boundary conditions considered in this calculation are therefore as follows:

$$Y_R'' = Y_D'' = 0 \quad \text{and} \quad Y_R = Y_D = 0 \quad \text{at the edge};$$

$$Y_R'''' = Y_D'''' = 0 \quad \text{and} \quad Y_R' = Y_D' = 0 \quad \text{at the center line.}$$

In Problem 1 the computations for steps 4 and 8 are initiated at the left edge where the values of  ${}_1Y_R''$ ,  $Y_D''$ ,  $Y_R'$ , and  $Y_D'$  are known to be zero. The computations for steps 3 and 7 are initiated at the center line where  ${}_1Y_R'''$ ,  $Y_D'''$ ,  $Y_R'$ , and  $Y_D'$  are known to be zero. The equivalent concentrations given for the center node point are the total concentrations of the function at the center line. The equivalent concentrations for the central node point are then split into two equal parts for determining the average value of the first integral in the segment adjacent to the central node point.

The value of  $K$  is found in this example by taking a ratio of sums as indicated. For this problem the exact solution is readily obtained from Bryan's formula, as  $P_{cr} = 39.48 N/b^2$ . The value obtained by the numerical procedure is  $39.46 N/b^2$ . Reasonable accuracy can be obtained with even larger segments in this problem. For two divisions in the plate width ( $n = 2$ ) the critical stress obtained by the numerical procedure is  $38.40 N/b^2$  and for three divisions ( $n = 3$ ) the critical stress obtained is  $39.28 N/b^2$ .



Taken from Fig. 5 of Reference (2)

Fig. 2. FORMULAS FOR EQUIVALENT CONCENTRATIONS.

Only a slight error in the assumed deflection configuration may be reflected in the comparison between  $Y_a$  and  $Y$ . Also round-off errors sometimes make an exact check difficult. Fortunately, the value of  $K$ , which is proportional to the buckling load, is not nearly as sensitive as the deflection configuration to small errors in the assumed configuration. The required measure of agreement of deflections seems to be a matter of judgment since it depends somewhat on the problem. However, for a given number of segments, the agreement between  $Y_a$  and  $Y$  can be improved to any degree desired by performing further cycles of calculations.

The application of the procedure to a more difficult problem is indicated in Problem 2 in a later section.

### ESTIMATE OF CRITICAL LOAD

In order to compare the computed values of deflection  $Y$  with the assumed values  $Y_a$ , it is first necessary to determine a value of  $k$ . The method of determining  $K$  should be such that a reasonable value is obtained even when the assumed deflection configuration is considerably in error. When the assumed deflection configuration is exact, the method must yield the proper value of  $K$  for the number of segments considered. Two methods are recommended.

The simplest and most convenient method of estimating  $K$  is to make the sum of the errors equal to zero. The error at a node point is defined to be the difference between the computed and assumed values of deflection. In this way Equation [8] is derived.

$$K = \frac{\sum (Y_a - Y_R)}{\sum Y_D} \quad [8]$$

A more reliable method utilizes the principle of least squares. In this method the value of  $K$  is chosen so that the sum of the squares of the errors is minimized. This procedure leads to Equation [9].

$$K = \frac{\sum (Y_a - Y_R) Y_D}{\sum Y_D^2} \quad [9]$$

For most problems the simple summation method is satisfactory and is preferable because it is more convenient than the least squares procedure. However, in problems where the deflection configuration is partly positive and partly negative, the least squares procedure is necessary because the summation method is not accurate. In general, when the summation method proves to be unsatisfactory, the least squares method should be used.

### BOUNDARY TREATMENT

The numerical procedure will progress in a straightforward manner if the boundary conditions are such that a value of each of  $Y'''$ ,  $Y''$ ,  $Y'$  and  $Y$  is prescribed in advance at some point or points. This is the case, for example, in Problem 1 where  $Y'''$  and  $Y'$  are known to be zero at the axis of symmetry and  $Y''$  and  $Y$  are known to be zero at the hinged edge. In general, other boundary conditions require an additional computation.

When these initial values are not all known in advance, a boundary value is assumed where needed to continue the integration procedure. Then at some later integration an inconsistent boundary value will be encountered, unless, by chance, the previously mentioned assumption was correct. If a boundary value is inconsistent, a correction may be made by finding the effect on this boundary value of a unit change in the earlier assumed value, and adopting as much of this unit correction as is necessary to satisfy the boundary requirement consistently.

This procedure is illustrated by Problem 2 which presents the solution of the problem of the buckling of a rectangular plate subjected to a linearly varying longitudinal force. The two transverse edges are hinged; one longitudinal edge is fixed, and the other is free. The boundary conditions are as follows:

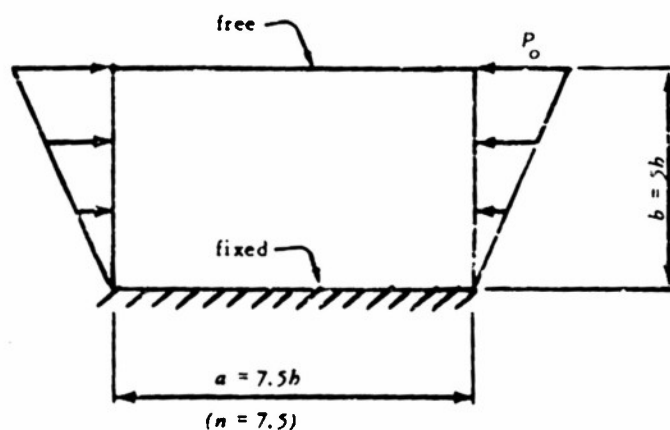
$$Y' = 0 \quad \text{and} \quad Y = 0 \quad \text{at the fixed edge;}$$

$$Y' = 3.257 h^2 Y''' \quad \text{and} \quad Y = 22.80 b^2 Y'' \quad \text{at the free edge.}$$

The numerical work is straightforward and follows Fig. 1 and Problem 1 except where explained here. When values of  $Y'''$  are to be determined, since no boundary values are available, a trial value is assumed and the procedure is continued. Similarly a trial value of  $Y''$  must be assumed. The boundary values at the fixed edge are used to initiate the calculations for the values of Trial  $Y'$  and Trial  $Y$ .



# PROBLEM 2



$$\frac{\pi^2}{n^2} = 0.1755$$

$$2 \frac{\pi^2}{n^2} = 0.3509$$

$$\frac{\pi^4}{n^4} = 0.0308$$

## Free Edge Requirements

$$Y' = 3.257 h^2 Y'''$$

$$Y = 22.80 h^2 Y''$$

$$\text{Poisson's Ratio} = 1/4$$

Description											Common Factors
				First Correction Function							
$\Delta Y_1'''$		1.00		1.00		1.00		1.00		1.00	$c/h^3$
$\Delta Y_1''$	-5.00		-4.00		-3.00		-2.00		-1.00		$c/h^2$
$\Delta Y_1'$	-2.33		-4.00		-3.00		-2.00		-1.00	-0.17	$c/h$
$\Delta Y_1$	0	-2.33		-6.33		-9.33		-11.33		-12.33	$c/h$
$\Delta Y_1$	0		-2.33		-8.67		-18.00		-29.33		$c$
				Second Correction Function							
$\Delta Y_2'''$		1.00		1.00		1.00		1.00		1.00	$c/h^3$
$\Delta Y_2''$		0.50		1.00		1.00		1.00		0.50	$c/h^2$
$\Delta Y_2'$	0	0.50		1.50		2.50		3.50		4.50	$c/h$
$\Delta Y_2$	0		0.50		2.00		4.50		8.00		$c$

When we denote

$x_1$  = required amount of first correction function

and  $x_2$  = required amount of second correction function,

the following equations express the free edge requirements.

$$Y' + x_1 \Delta Y_1' + x_2 \Delta Y_2' = 3.257 h^2 (Y''' + x_1 \Delta Y_1''')$$

$$Y + x_1 \Delta Y_1 + x_2 \Delta Y_2 = 22.80 h^2 (Y'' + x_1 \Delta Y_1'' + x_2 \Delta Y_2'')$$

The values of  $Y'''$ ,  $Y''$ ,  $Y'$ , and  $Y$  above are denoted as "trial" values in the subsequent computations. Substituting the values of the correction functions above, one obtains the following equations.

$$Y' - 12.50 \frac{c}{h} x_1 + 5 \frac{c}{h} x_2 = 3.257 h^2 (Y''' + \frac{c}{h^2} x_1)$$

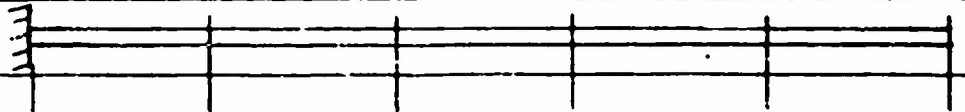
$$Y - 41.67 c x_1 + 12.50 c x_2 = 22.80 h^2 (Y'' + \frac{c}{h^2} x_2)$$

Solving for  $x_1$  and  $x_2$ , one determines the following relations.

$$74.13 x_1 = \frac{1}{c} (Y - 22.80 h^2 Y'') + 2.060 \frac{h}{c} (Y' - 3.257 h^2 Y''')$$

$$23.52 x_2 = \frac{1}{c} (Y - 22.80 h^2 Y'') - 2.644 \frac{h}{c} (Y' - 3.257 h^2 Y''')$$

PROBLEM 2 (continued)

Description								Common Factors
$Y_a$	0	1.7	5.6	10.6	15.8	21.2	$c$	
at free end $Y_a' = 5.7 c/h$ (calculated from previous cycle).								
${}_1Y_R''''$	0	-0.052	-0.172	-0.326	-0.487	-0.653	$c/h^4$	
${}_1\bar{Y}_R''''$		-0.058	-0.175	-0.327	-0.487	-0.299	$c/h^3$	
Trial ${}_1Y_R'''$	-1.154	-1.212	-1.387	-1.714	-2.201	-2.500	$c/h^3$	
Trial ${}_1Y_R''$	0.068	-1.086	-2.298	-3.685	-5.399	-7.600	$c/h^3$	
${}_2Y_R''$	0	0.596	1.965	3.720	5.544	7.439	$c/h^3$	
Trial $Y_R''$	0.068	-0.490	-0.333	0.035	0.145	-0.161	$c/h^3$	
Trial $\bar{Y}_R''$	-0.089	-0.430	-0.315	0.014	0.110	-0.012	$c/h$	
Trial $Y_R'$	0	-0.089	-0.519	-0.834	-0.820	-0.710	$c/h$	
Trial $Y_R$	0	-0.089	-0.608	-1.442	-2.262	-2.972	$c$	
at the free edge								
$Y_R - 22.80 h^2 Y_R'' = -2.972 c + (22.80) (0.161 c) = 0.699 c$								
$Y_R''' = {}_1Y_R''' + 2 \frac{\pi^2}{n^2} \frac{I}{h^3} Y_a' = -2.500 \frac{c}{h^3} + (0.3509) \left( 5.7 \frac{c}{h^3} \right) = -0.500 \frac{c}{h^3}$								
$Y_R' - 3.257 h^2 Y_R''' = -0.722 \frac{c}{h} + (3.257) \left( 0.500 \frac{c}{h} \right) = 0.906 \frac{c}{h}$								
Therefore								
$x_1 = 0.0346$ and $x_2 = -0.0721$								
$Y_R = \text{Trial } Y_R$ $+ 0.0346 \Delta Y_1$ $- 0.0721 \Delta Y_2$	0	-0.21	-1.05	-2.39	-3.85	-5.31	$c$	
at free end $Y_R' = \text{Trial } Y_R' + 0.0346 \Delta Y_1' - 0.0721 \Delta Y_2' = -1.52 c/h$								
Calculation of $Y_D$								
$p_x$	0	0.20	0.40	0.60	0.80	1.00		
${}_1Y_D'''' = Y_D''''$	0	0.06	0.39	1.12	2.22	3.72	$c/h^4$	
$\bar{Y}_D''''$		0.08	0.42	1.15	2.25	1.59	$c/h^3$	
Trial $Y_D'''$	-2.24	-2.16	-1.74	-0.59	1.66	3.25	$c/h^3$	
Trial $Y_D''$	6.77	4.53	2.37	0.63	0.04	1.70	$c/h^2$	
Trial $\bar{Y}_D''$	3.01	4.54	2.40	0.73	0.23	0.48	$c/h$	
Trial $Y_D'$	0	3.01	7.55	9.95	10.68	10.91	$c/h$	
Trial $Y_D$	0	3.01	10.56	20.51	31.19	42.10	$c$	
At the free edge								
$Y_D - 22.80 h^2 Y_D'' = 42.10 c - (22.80) (1.70 c) = 3.34 c$								
$Y_D' - 3.257 h^2 Y_D''' = 11.39 \frac{c}{h} - (3.257) \left( 3.25 \frac{c}{h} \right) = 0.80 \frac{c}{h}$								

# PROBLEM 2 (continued)

	Therefore						
	$x_1 = 0.0673$ and $x_2 = 0.0521$						
$Y_D = \text{Trial } Y_D$	0	2.88	10.08	19.53	29.63	39.95	c
$+0.0673 \Delta Y_1$							
$+0.0521 \Delta Y_2$							
	At the free end						
	$Y_D' = \text{Trial } Y_D' + 0.0673 \Delta Y_1' + 0.0521 \Delta Y_2' = 10.81 \text{ c/h}$						
	$K = \frac{\sum(Y_a - Y_R)}{\sum Y_D} = \frac{67.71}{102.07} = 0.663$						
$Y = Y_R + KY_D$	0	1.7	5.6	10.6	15.8	21.2	c
	At the free end						
	$Y' = Y_R' + KY_D' = 5.6 \text{ c/h}$						
$(P_o)_{cr} = 0.663 \frac{N}{b^2} = 16.6 \frac{N}{b^2}$							

The boundary requirements at the free edge are not initially satisfied; therefore the assumed values of  $Y'''$  and  $Y''$  are not correct. Note that at the free edge the boundary value of  $Y'$  is a function of  $Y'''$ . However, only  $Y_R'''$  is given in the tabulated computations. In this case  $Y_R'''$  is computed from Equation 7(a), which in turn requires the slope of the assumed deflection curve at the free edge to be known. Therefore, when one of the boundaries is a free edge the slope  $Y_a'$  at that edge must also be assumed and the final computed slope must agree with this value, just as the assumed and computed deflections must agree in order for the solution to be valid. It is usually more accurate to take the slope of the curve at the boundary from the preceding computation rather than to use an interpolation formula. For this case the difference is slight since the slope  $Y_a'$  found by the use of either a three point or four point approximating polynomial is  $5.5 \text{ c/h}$  as compared with the value of  $5.7 \text{ c/h}$  found from the previous cycle. The latter is more accurate.

Two correction configurations are required, one for each assumption. Each correction configuration influences both boundary requirements. The correct proportions of the first and second correction configurations ( $\Delta Y_1$  and  $\Delta Y_2$ , respectively) may be determined from the equations presented just below the computations for the configurations.

The values of the required corrections are added to the values of Trial  $Y$  to yield the correct deflections; similarly the correct end slopes are found. Both  $Y_R$  and  $Y_D$  are corrected independently of each other.

The value of  $K$  is found to be 0.663, which gives a value of the critical buckling stress (at the free edge) of  $16.6 \text{ N/b}^2$ .

## CONVERGENCE

In most problems the resulting deflection configuration  $Y$  is a better approximation to the true buckling configuration than the assumed deflection shape  $Y_a$ . In this case the process is said to converge toward the true solution. On the other hand, in limited classes of problems  $Y$  may be further than  $Y_a$  from the true solution, and in this case the process is divergent. A problem in which the process is convergent may be handled most simply by using the resultant deflection configuration  $Y$  of the last cycle for the assumed configuration  $Y_a$  of the next cycle. This obvious procedure will not be efficient when the process converges slowly; in this case extrapolation as described below is desirable. When the process diverges, an extrapolation may be used to obtain an answer in many cases. In others, no solution is possible with successive approximation techniques of this type.

It is not a simple matter to determine in advance whether or not the successive approximations converge. However, from experience in solving a variety of problems it may be stated qualitatively that if there are tensile forces in the plate, the larger the amount of tensile force the greater will be the tendency to diverge. It may also be stated that if divergence occurs, the successive solutions will generally oscillate about the true solution. These features are similar to those which apply in other types of buckling problems in which successive approximation techniques are used.

The recommended extrapolation procedure<sup>4</sup> is applied as follows: Let successive deflections at any point be designated as  $A$ ,  $B$ , and  $C$ , respectively, where  $A$  is the first assumption,  $B$  follows from  $A$ , and  $C$  from  $B$ . Then, the extrapolated value  $\bar{A}$  is:

$$\bar{A} = \frac{AC - B^2}{A + C - 2B}$$

<sup>4</sup>See "Bounds and Convergence of Relaxation and Iteration Procedures," by N. M. Newmark, these Proceedings.

If this formula is used, it must be applied separately to the deflections at each point. Thus, after two steps in the calculation, which yield values of  $B$  and  $C$  at all points corresponding to assumed values  $A$ , one computes the values of  $A$ , and repeats the calculation. Ordinarily one may have to go through this operation several times in problems which diverge rapidly.

Problems in which divergence occurs present a difficulty which becomes immediately obvious to the computer. He may have to use other techniques to solve the problem if the successive approximations procedure does not lead to an answer either directly or with the extrapolation procedure described.

### SUMMARY

The procedure described is useful for a wide variety of loadings. With some modifications the problem of the buckling of ribbed plates is easily solved. In addition, many other practical problems which do not satisfy all of the requirements of this method can be treated by either simplifying the problem or by waiving some of the limita-

tions of the method which may not be important for a particular case. Extension of the procedure to problems which do not satisfy the limitations of the method requires engineering judgment.

The results are approximate but can be made as accurate as is desired by taking a sufficient number of divisions in the plate width. As the number of divisions is increased the accuracy of the solution increases and also the amount of work required to effect a solution increases. The question of how fine a network to use depends somewhat on the problem and can best be decided by experience. Ordinarily, however, for most practical problems no more than five or six divisions in the plate width are needed.

### ACKNOWLEDGMENTS

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A METHOD FOR THE SOLUTION  
OF THE RESTRAINED CYLINDER  
UNDER COMPRESSION

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by E. D'Appolonia  
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## A METHOD FOR THE SOLUTION OF THE RESTRAINED CYLINDER UNDER COMPRESSION

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### INTRODUCTION

A STRESS solution of most radially symmetric problems is difficult and tedious by formal mathematical procedures. In this work a numerical method is developed that can be readily applied to solve such problems. The particular problem selected to illustrate the method is that of the restrained cylinder under compression. The method requires a framework of pin-connected bars which, for similar load conditions, will approximate the deformation of the solid body. This method of solving stress problems is referred to as the lattice analogy method. The arrangement of the bars in any element of the framework is arbitrary. Normally a pattern that is simple and suitable for the type of problems being considered is adopted. It is apparent that the smaller the size of the mesh of the framework the better the approximation of the deformation of the solid body by the framework.

To determine the deformation of the framework it is necessary to know the area of each bar. These areas are derived from the requirement that the distortion of an element of the solid body and of the framework be the same for similar load conditions. Once the bar areas are established the relationship between the force and the end displacements of each bar can be formulated. Then from the equilibrium conditions of the joints, displacement equations are developed. In these equations a component displacement of a central joint is related to the two component movements of the immediate surrounding joints. The set of simultaneous equations that arise in a particular problem is solved either by an iteration or relaxation technique. For the problems solved in this work an iteration process was used.

The advantages gained in the use of a lattice analogy method are often overlooked. Once the size of the mesh of the framework has been selected the only approximation involved in a solution of a problem is the degree of precision to which the numerical work is carried. With the use of finite differences besides the approximations due to the size of the mesh and the precision of the nu-

merical work, there is the approximation of the differential equation by differences. Also, the difficulties that arise at boundaries with a finite difference procedure for example, at a right angle or re-entrant corner are not present in the lattice analogy procedures. The framework also lends itself to the solution of elastic-plastic problems.

As with a finite difference procedure where the solution of a problem can be set forth either in terms of a stress function or in terms of displacements, the lattice analogy method can be formulated in terms of bar forces (1), (2)\* or in terms of displacements (3), (4). The method developed in this paper deals with displacements. The main advantages in using displacements are (1) the boundary conditions can be prescribed either in terms of displacements or forces (2) the body can be either singly or multi-connected. The numerical displacement procedure of analysis appears to offer the only means of solving stress problems for which the body is multi-connected and the boundary conditions are prescribed forces and displacements.

The assumptions made in the classical theory of linear elasticity apply to this work.

### NOMENCLATURE

The following nomenclature is used in this paper:

- $r, \theta, z$  = cylindrical coordinates
- $\sigma_r, \sigma_\theta, \sigma_z$  = normal stresses in the  $r, \theta, z$  directions
- $\tau_{rz}$  = shearing stress in the  $r-z$  plane
- $\epsilon_r, \epsilon_\theta, \epsilon_z$  = normal strains in the  $r, \theta, z$  directions
- $\nu_{rz}$  = detrusion in the  $r-z$  plane
- $\epsilon = \epsilon_r + \epsilon_\theta + \epsilon_z$  = dilatation
- $E$  = modulus of elasticity
- $G$  = shear modulus
- $\mu$  = Poisson's ratio
- $u, w$  = radial and axial displacement components
- $A$  = area of a bar
- $e$  = strain in a bar
- $s$  = stress in a bar

\*Numbers in parentheses refer to references listed in the bibliography at the end of the paper.

$F$  = force in a bar

$\Lambda$  = length parameter

$n$  = integer

$R, Z$  = normal force concentrations at a joint

$S$  = shearing force concentrations at a joint

$a, \alpha, \Lambda_0$  = arbitrary constants used with expressions for the bar areas

$c, F, m$  = arbitrary constants used with the "second-order" force systems

$$\sigma_r = \frac{\sqrt{2}}{2} \left[ (\sigma_r - \sigma_\theta)^2 + (\sigma_\theta - \sigma_z)^2 + (\sigma_z - \sigma_r)^2 + 6\tau_{rz}^2 \right]^{1/2}$$

= intensity of stress

### THEORY

The stress solution for a radially symmetric body is obtained most conveniently with reference to a system of cylindrical coordinates. A point  $P(r, \theta, z)$  is defined by its radial, tangential and axial coordinate respectively. The radial and axial displacements of the distorted body are designated by  $u$  and  $w$ , respectively. The normal stress components designated by  $\sigma_r, \sigma_\theta$ , and  $\sigma_z$  act in the radial, tangential and axial direction and the shear stress  $\tau_{rz}$  acts in the  $r$ - $z$  plane. The remaining stress components, due to symmetry, are zero. Based on the assumption of small displacements the normal strain components and detrusion are respectively

$$\epsilon_r = \frac{\partial u}{\partial r}, \quad \epsilon_\theta = \frac{u}{r}, \quad \epsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

For an elastic material which obeys Hooke's Law the stress strain relations are given by

$$\sigma_r = \frac{E}{(1+\mu)(1-2\mu)} \left[ (1-\mu)\epsilon_r + \mu(\epsilon_\theta + \epsilon_z) \right]$$

$$\sigma_\theta = \frac{E}{(1+\mu)(1-2\mu)} \left[ (1-\mu)\epsilon_\theta + \mu(\epsilon_r + \epsilon_z) \right]$$

$$\sigma_z = \frac{E}{(1+\mu)(1-2\mu)} \left[ (1-\mu)\epsilon_z + \mu(\epsilon_r + \epsilon_\theta) \right]$$

$$\tau_{rz} = \frac{E}{1+\mu} \gamma_{rz}$$

The solution of a particular problem requires either a determination of the stress components or the displacement components. For example, it can be readily shown from a consideration of (1), and (2) and the equations of equilibrium

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = 0$$

that the differential equations to be satisfied, subject to boundary conditions, by the displacement components are

$$\Delta^2 u - \frac{1}{r^2} + \frac{1}{1-2\mu} \frac{\partial \epsilon}{\partial r} = 0$$

$$\Delta^2 w + \frac{1}{1-2\mu} \frac{\partial \epsilon}{\partial z} = 0$$
[3]

However, with either the differential equations, Equation [3], or those in terms of stress components a solution of a particular problem is intractable by formal mathematical procedures and an approximate numerical method of analysis has to be resorted to. Such a method is developed in the following work. Since it involves displacements it can be considered as the analog of the integration procedure that is necessary to integrate equations [3].

A suitable pattern of bars is shown in Fig. 1. In an axial plane the framework is built-up of square lattices with crossed-diagonals. A cross-section is composed only of radial and tangential bars. To establish the bar areas, the stress systems required to produce identical deformations of the framework and solid body are considered. Since the deformation of an element can be described fundamentally in terms of three strain components, the deformations considered are those associated with a uniform strain in the radial and axial direction and a detrusion. Thus for the solid body a uniform strain in the axial direction of

$$\epsilon_z = 1$$

$$\epsilon_r = \epsilon_\theta = \gamma_{rz} = 0$$

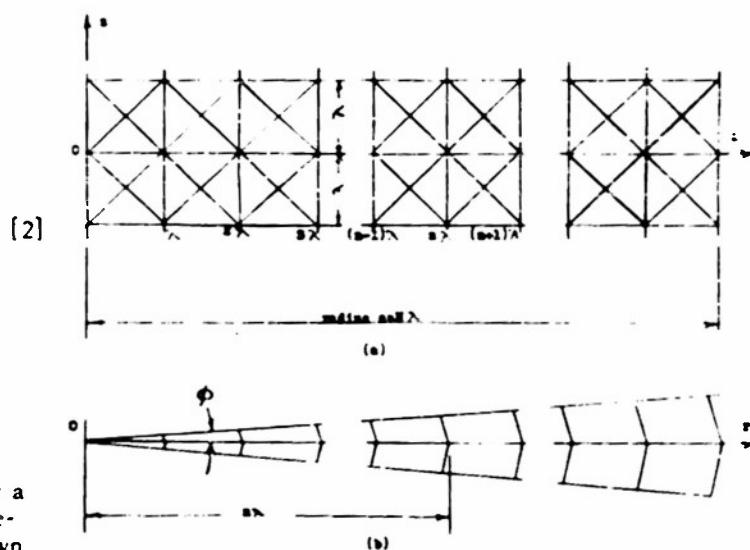


Fig. 1. FRAMEWORK OF BARS.

is produced by the stress distributions

$$\sigma_r = \sigma_\theta = \frac{\mu E}{(1 + \mu)(1 - 2\mu)}$$

$$\sigma_z = \frac{(1 - \mu)E}{(1 - \mu)(1 - 2\mu)}$$

$$\tau_{rz} = 0$$

A uniform strain in the radial direction that is

$$\epsilon_r = \epsilon_\theta = 1$$

$$\epsilon_z = \gamma_{rz} = 0$$

requires the stress distribution

$$\sigma_r = \sigma_\theta = \frac{E}{(1 + \mu)(1 - 2\mu)}$$

$$\sigma_z = \frac{2\mu E}{(1 + \mu)(1 - 2\mu)}$$

$$\tau_{rz} = 0$$

For a detrusion

$$\frac{\gamma_{rz}}{2} = \frac{1}{r}$$

the stresses

$$\tau_{rz} = \frac{1}{1 + \mu} \cdot \frac{1}{r}$$

$$\sigma_r = \sigma_\theta = \sigma_z = 0$$

are required.

Before considering states of deformation similar to the foregoing for the framework, it is expedient to choose the form of the expression for the area of each bar. It is assumed that the area of each bar is proportional to the corresponding cross-sectional area of a solid element with height  $\lambda$ , radial length  $\lambda$  and average width  $(n + 1/2)\lambda\phi$ . For example, the area of the axial bar is proportional to the axial cross-sectional area  $\eta\lambda^2\phi$  of the solid element, and likewise for the remaining bars. Hence the areas for the radial, axial, tangential and diagonal bars are

$$A_r = \left(n + \frac{1}{2}\right)\lambda^2\phi A_0$$

$$A_z = n\lambda^2\phi A_0$$

$$A_\theta = a\lambda^2\phi A_0$$

$$A_d = a\left(n + \frac{1}{2}\right)\frac{\lambda^2\phi}{\sqrt{2}} A_0$$

where  $A_0$ ,  $a$  and  $\phi$  are arbitrary constants  $\lambda$  and  $\phi$  are defined in Fig. 1 and  $n$  is an integer. All the areas with the exception of those of the tangential bars are dependent on the radial distance.

Other selections for the bar areas are possible, however, with this selection the resulting bar areas and displacement equations to be developed, take on simple expressions convenient for numerical computation.

Because of the selected bar pattern and areas it was found necessary to introduce into the force system for each type of deformation a "secondary" system of bar forces. Without the use of these "secondary" systems the coefficients  $A_0$ ,  $a$  and  $\phi$  in Equations [7] would turn out to be functions of radial distance, thus complicating the form of the displacement equations. Further, the use of such systems with a square lattice having crossed-diagonals permitted a determination of the bar areas in terms of an arbitrary value of Poisson's ratio. In previous work, [4], secondary bar patterns had to be introduced when Poisson's ratio differed from  $1/3$ . The use of secondary bar patterns leads to complications and is to be avoided.

Without loss of generality and to avoid carrying unnecessary constants  $E$ ,  $\lambda$  and  $\phi$  are taken as unity hereafter.

Considering the framework, the forces due to a uniform axial strain are shown in Fig. 2. Component forces  $maA_0$  and  $qmaA_0$  are the resultant radial components of the forces in the tangential bars. The forces in the radial and tangential bars given by  $k(n + 1/2)A_0$  and  $maA_0$  are due to the "secondary" force system associated with a uniform axial strain. From the statics of joints  $n$  and  $e$ , Fig. 2, the equivalent uniform stresses are

$$s_z = \left(\frac{a}{2} + 1\right) A_0$$

$$s_r = \frac{\left[\left(k + \frac{a}{2}\right)n + \frac{1}{2}\left(k + \frac{a}{2}\right) + qma\right] A_0}{n + 1}$$

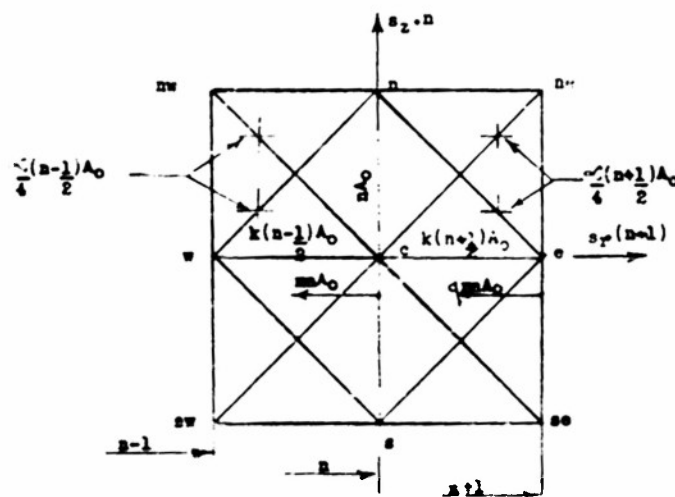


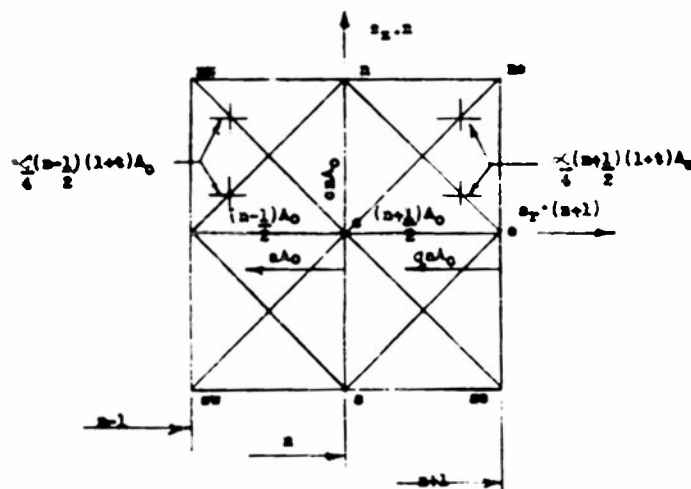
FIG. 2. BAR FORCES DUE TO UNIFORM AXIAL STRAIN.



$$ma = k \cdot \frac{a}{2} \quad (9)$$

$$s_z = \left[ c + \frac{a}{2} (1 + t) \right] A_o$$

$$s_r = \frac{\left[ n + \frac{l}{2} qa + \frac{a}{2} \left( n + \frac{l}{2} \right) (1 + t) \right] A_o}{n + l} \quad (10)$$



At joint c the sum of the radial forces gives

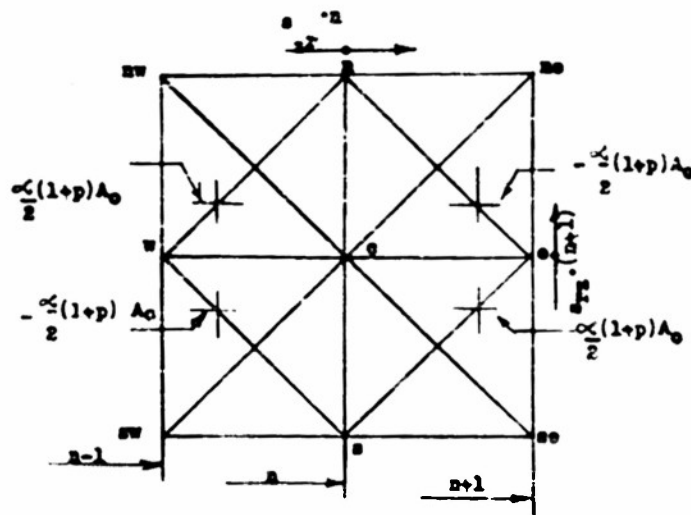
$$a = 1 + \frac{a}{2}(1 + t) \quad [11]$$

Corresponding to a detrusion of  $\frac{\gamma_{yz}}{2} = \frac{l}{r}$  in the solid body the strain in the diagonal bars is

$$e_d = \frac{l}{n}$$

and all other bar strains are zero. The bar forces for this case are shown in Fig. 4. The equilibrium of joints  $n$  and  $e$  requires that

$$S_{n+1} = S_n (n+1) = a(1+p) A_n \quad [12]$$



The force  $\pm \frac{\sqrt{2}}{2} \rho A_o$  is to be associated with the "secondary" force system.

In the above work the constants  $k$ ,  $m$ ,  $q$ ,  $c$ ,  $t$ , and  $p$  are to be determined along with  $a$ ,  $\alpha$  and  $A_0$ . These constants are evaluated by equating corresponding equations of stress for similar states of deformation in the solid body and framework. Hence from Equations [4] and [8], [5] and [10], [6] and [12] and simplifying by means of Equations [9] and [11] and taking  $q = 1/2$ , one obtains the following expressions for the constants:

$$\lambda_o = \frac{2(1-\mu)}{(1+\mu)(1-2\mu)(2+a)}$$

$$k = \frac{2\mu(1+a) - a}{2(1-\mu)}$$

$$l = \frac{2k}{a} + 1$$

$$a = l + k + \alpha \quad [13]$$

$$ma = k + \frac{a}{2}$$

$$c = k$$

$$p = \frac{1 - k - a}{a}$$

Since a total of nine constants are to be determined from the above seven expressions two of them can be chosen arbitrarily. For the work to follow a value of  $1/4$  is taken for Poisson's ratio. For simplicity of results  $a$  was taken as one. Accordingly, the values for the constants are  $k = p = c = 0$ ,  $a = 2$ ,  $t = 1$ ,  $m = 1/4$ ,  $A_0 = 4/5$  and as previously assigned,  $q = 1/2$ .

With these results the bar areas become

$$A_r = \frac{4}{5} \left( n + \frac{1}{2} \right)$$

$$A_z = \frac{4}{5} n$$

$$A_\theta = \frac{8}{5}$$

$$A_d = \frac{4}{5\sqrt{2}} \left( n + \frac{1}{2} \right)$$

It is interesting to note that for plane strain problems ( $\mu = 1/4$ ) the bar areas of a framework of square lattices with crossed-diagonals would be  $\frac{4}{5}$  and  $\frac{4}{5\sqrt{2}}$  for the horizontal and vertical bars and the diagonal bars respectively. For convenience and summary the bar forces and stresses for each type of deformation considered in the development of the bar areas are given in Figs. 5, 6 and 7.

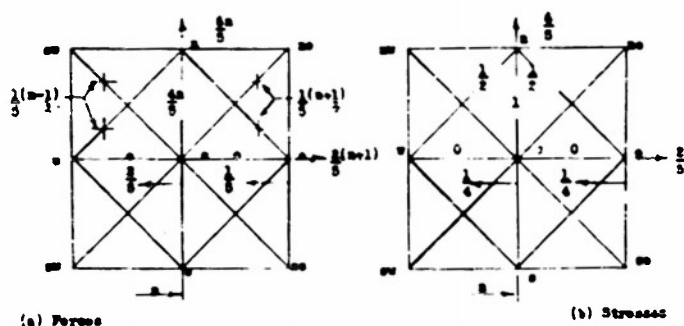


Fig. 5. UNIFORM AXIAL STRAIN,  $e_z = i$ .

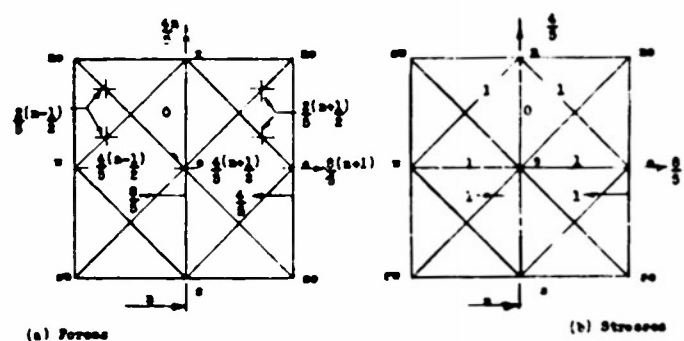


Fig. 6. UNIFORM RADIAL STRAIN,  $e_r = i$ .

From Fig. 5 it can be seen that each unit of axial strain requires, besides the expected bar stresses, a "secondary" stress of  $1/4$  in the tangential bars to maintain radial equilibrium. Likewise, Fig. 6 shows a "secondary" stress of  $1/2$  is required in the diagonal bars. In developing the displacement equations these "secondary" stress systems are taken into account.

The area of the bars at the boundaries of the frame-

[14]

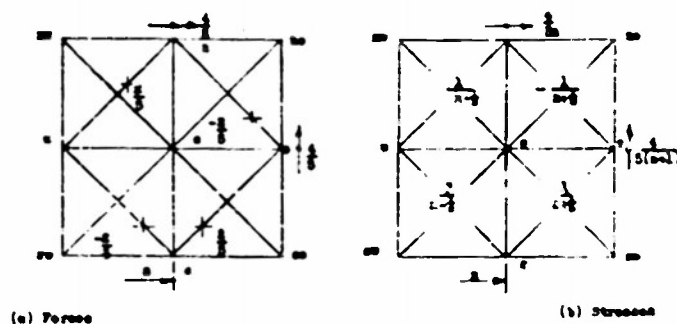


Fig. 7. DETRUSION  $e_d = 1/n$ .

work are fractional parts of the corresponding interior bar areas. Thus the radial and tangential bars in a radial plane have one-half the area of the same interior bar. At a corner the tangential bar area is one-quarter of the same interior bar. At an axial boundary a distance  $n\lambda$  from the origin the axial bar has an area of  $\frac{2}{5}(n - 1/2)$ . This is consistent with the area  $4/5 n$  of the interior bar at a distance  $n\lambda$  from the origin since it is considered that the area  $4/5 n$  is made up of two bars with areas  $\frac{2}{5}(n - \frac{1}{2})$  and  $\frac{2}{5}(n + 1/2)$ . From similar reasoning, the area of each axial bar at  $n\lambda = 0$  is  $1/5$ .

Having established values for the bar areas and stress-strain relations, shown diagrammatically in Figs. 5, 6 and 7, one can proceed to a derivation of the displacement equations. For a general interior joint as shown in Fig. 8 forces, stresses, etc., of all bars radiating from joint  $c$  are denoted by the letter or letters of the joint remote from  $c$ . For example  $s_e$  stands for the stress in bar  $c-e$ , or  $F_{ne}$  is the force in bar  $c-ne$ . However, the symbols  $u$  and  $w$

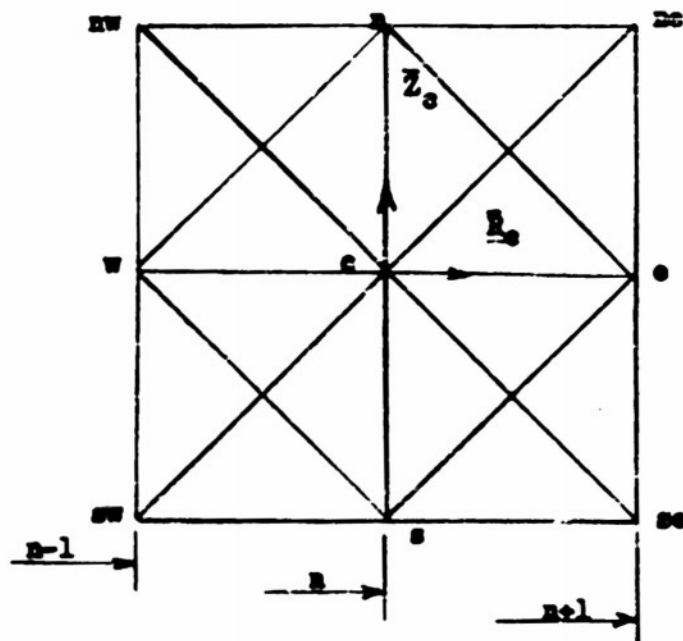


Fig. 8. GENERAL INTERIOR JOINT.

refer to displacements at a joint so that  $u_{se}$  implies the radial displacement of joint  $se$ . Considering then a deformed framework and using the stress strain relations derived one can write for the bar stresses

$$s_e = u_e - u_c, \quad s_w = u_c - u_w$$

$$s_n = w_n - w_c, \quad s_s = w_c - w_s$$

$$s_{ct} = \frac{u_c}{n} + \frac{w_n - w_s}{8}$$

$$s_{se} = \frac{u_{se} - u_c}{2} + \frac{1}{2}(w_c - w_{se}) \quad [15]$$

$$s_{ne} = \frac{u_{ne} - u_c}{2} + \frac{1}{2}(w_{ne} - w_c)$$

$$s_{sw} = \frac{u_c - u_{sw}}{2} + \frac{1}{2}(w_c - w_{sw})$$

$$s_{nw} = \frac{u_c - u_{nw}}{2} + \frac{1}{2}(w_{nw} - w_c)$$

The equilibrium of joint  $c$ , Fig. 7 requires

$$\bar{R}_c + F_e - F_w - F_{ct} + \frac{l}{\sqrt{2}}(F_{ne} + F_{se} - F_{nw} - F_{sw}) = 0 \quad [16]$$

$$\bar{Z}_c + F_n - F_s + \frac{l}{\sqrt{2}}(F_{ne} + F_{nw} - F_{se} - F_{sw}) = 0$$

where  $\bar{R}_c$  and  $\bar{Z}_c$  are the component body forces acting at  $c$ . Multiplying the expressions in Equations [15] by their appropriate area as given by Equation [14] and substituting into Equations [16] and simplifying, one obtains for the radial and axial displacement equations of the joint  $c$ ,

$$u_c = \frac{n}{2(2n^2 + 1)} \left[ \frac{\bar{R}_c}{A_o} + \left(n + \frac{1}{2}\right) u_e + \left(n - \frac{1}{2}\right) u_w - \frac{w_n - w_s}{4} + \frac{l}{2} \left(n + \frac{1}{2}\right) \left(u_{ne} + \frac{w_{ne}}{2} + u_{se} - \frac{w_{se}}{2}\right) + \frac{l}{2} \left(n - \frac{1}{2}\right) \left(u_{nw} - \frac{w_{nw}}{2} + u_{sw} + \frac{w_{sw}}{2}\right) \right] \quad [17]$$

$$w_c = \frac{l}{3n} \left[ \frac{\bar{Z}_c}{A_o} + n(w_n + w_s) + \frac{l}{2} \left(n + \frac{1}{2}\right) \left(u_{ne} + \frac{w_{ne}}{2}\right) \right]$$

$$- u_{se} + \frac{w_{se}}{2} + \frac{l}{2} \left(n - \frac{1}{2}\right) \left(u_{sw} + \frac{w_{sw}}{2} - u_{nw} - \frac{w_{nw}}{2}\right) \Bigg]$$

In a manner similar to the above, displacement equations can be derived for the various boundary joints that occur in a problem.

To solve a particular problem one must first select the size of the mesh of the framework. Also, for that portion of the surface of the solid body subjected to stress a system of equivalent force concentrations must be placed on the boundary of the framework. The linear set of displacement equations that occur in a particular problem can best be solved by either an iteration or relaxation procedure. A discussion of such procedures or the details as to how a solution of a problem is carried out need not be given here.

From the determination of the joint displacements the force in each bar can be established by multiplying the appropriate stress equation, Equation [15], by its corresponding bar area, Equation [14]. The average uniform stress components at each joint are then obtained from a consideration of sections as shown in Figs. 9 and 10. That is for an axial section the average radial stress at joint  $c$  is

$$\sigma_r = \frac{R_c}{n}$$

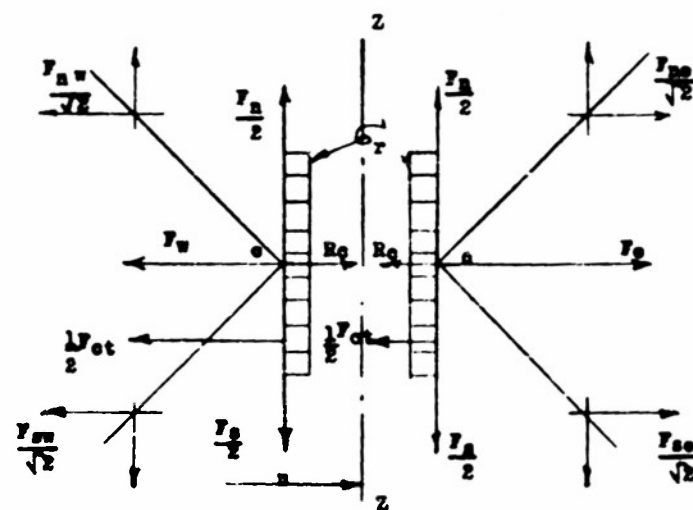


FIG. 9. SECTION Z-Z FOR EVALUATION OF RADIAL STRESS.

The tangential stress is

$$\sigma_\theta = \frac{F_{ct}}{l}$$

From a radial section, the average axial stress is given by



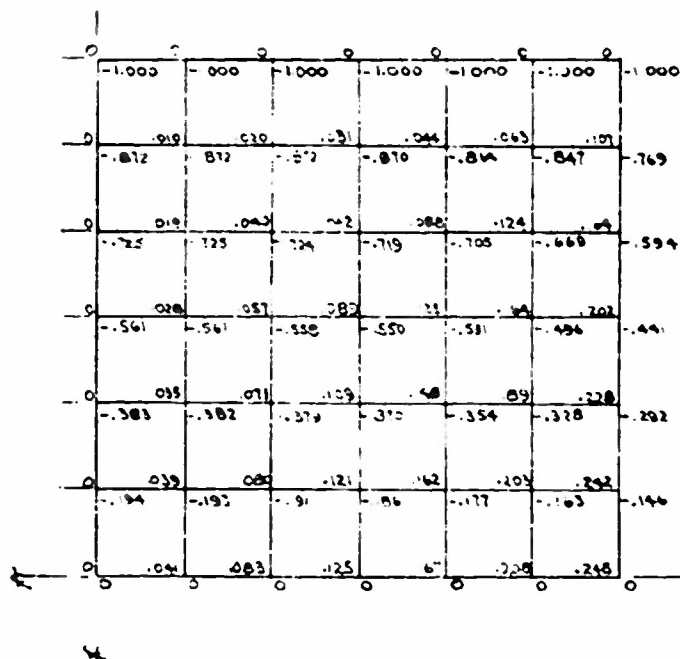


Fig. 12. DISPLACEMENTS FOR END-RESTRAINED CYLINDER

$E = 1, \mu = 1/4, \lambda = 1/2$ . AVERAGE COMPRESSIVE  
STRESS  $\sigma_c = -0.348$ .

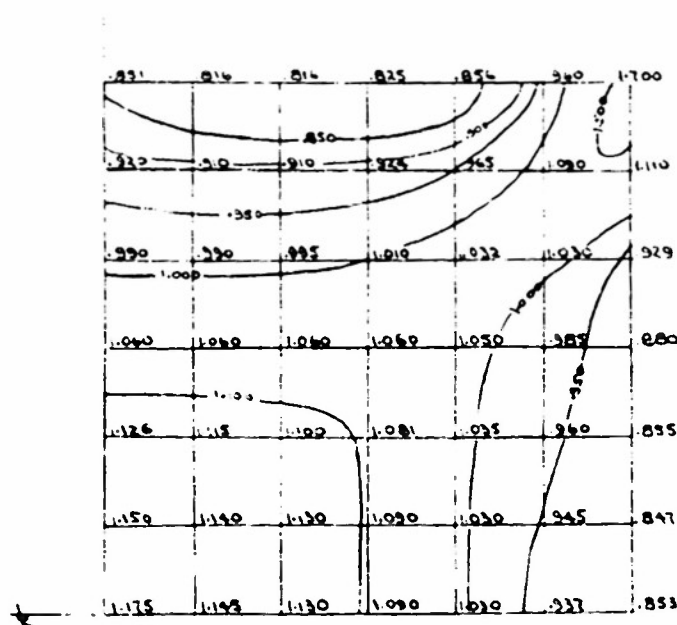


Fig. 13. RATIO OF AXIAL STRESS  $\sigma_z$  TO AVERAGE COMPRESSIVE STRESS  $\sigma_c = -0.348$ .

series solution for the stresses, Pickett was unable to establish maximum values for the stresses at the outer edges of contact. The maximum ratios of axial, radial, tangential and shear stress to average compressive stress determined by the lattice analogy method are 1.70, 0.64,

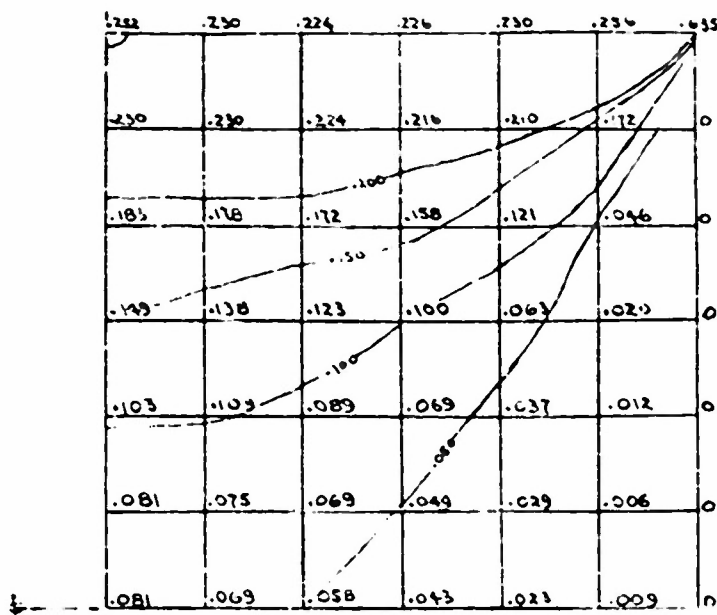


Fig. 14. RATIO OF RADIAL STRESS  $\sigma_r$  TO AVERAGE COMPRESSIVE STRESS  $\sigma_c = -0.348$ .

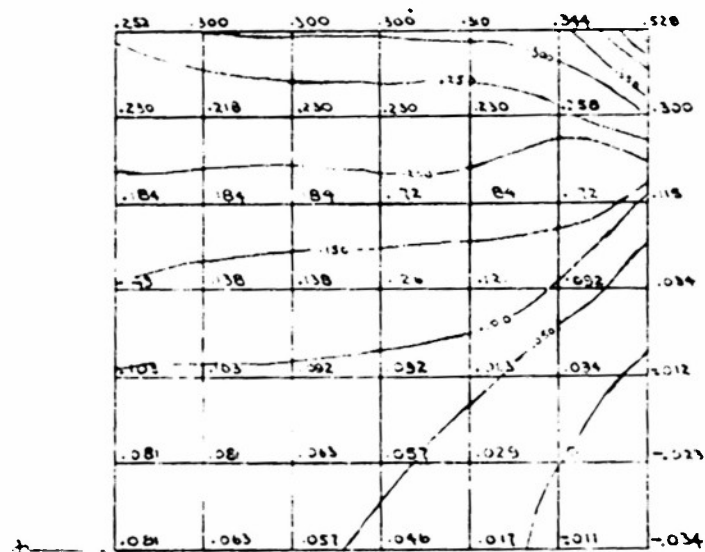


Fig. 15. RATIO OF TANGENTIAL STRESS  $\sigma_\theta$  TO AVERAGE COMPRESSIVE STRESS  $\sigma_c = -0.348$ .

0.53 and 0.66, respectively. It was estimated in (6) that the maximum value of  $\sigma_z$  would be less than  $\frac{1}{1-2\mu}$  or, for our case twice the average compressive stress. The lattice analogy method appears to support this estimate. Though a finer size mesh was not investigated it is felt



the outer edge of contact of the cylinder and rigid block, a large central portion of the cylinder would start to flow plastically.

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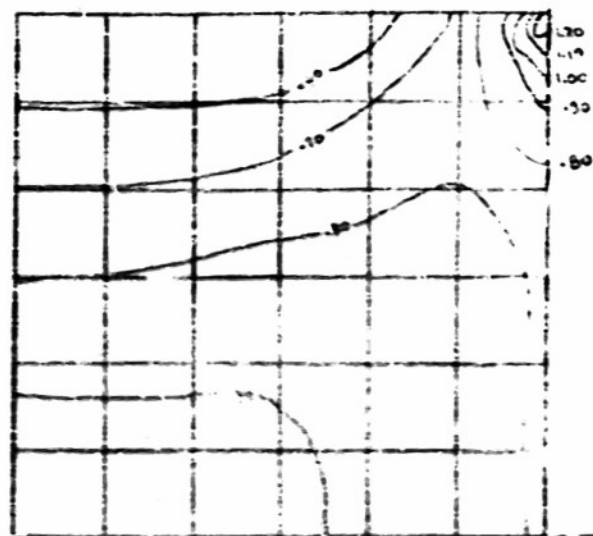


FIG. 10. PLOT OF RATIO OF INTENSITY OF STRESS  $\sigma$  TO SHEAR MODULUS  $G$ .